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GNOSTICAL THEORY OF INDIVIDUAL DATA

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Theory of a new approach to the problem of data treatment is presented. In contrast to the statistical approach the presented "gnostical" theory is based on a detailed study of effects of uncertainty on individual data. Geometrical features of the space mapping the real data are derived from a simple axiom. Quantitative characteristics of the uncertainty are shown to satisfy an equation of a diffusion type from which a formula of information borne by individual datum is derived. Each individual datum determines an ideal closed cycle of gnostical transformations minimizing the unavoidable loss of information caused by the uncertainty.

1. Introduction

This paper deals with a mathematical model of randomness, applicable to individual uncertain events and to small samples of real data. The aim is very practical: to derive efficient and robust formulae and algorithms for data processing applicable when a statistical model of the uncertainty is not known or not justified. The idea is: "Let data speak for themselves!" But to achieve such a goal it would be necessary to have a theory which would offer another insight under the surface of uncertain events than statistics does. The author's belief is that this problem can be solved but not by mathematical tools commonly used in statistics and probability theory. The appropriate mathematics is very old, it is Riemannian geometry, developed already in XIXth century and applied with a brilliant success in physics three quarters of a century ago. This paper is an attempt to show that Riemannian–Einsteinian idea on necessary dependence of metrics of spaces on some facts of real life is fully relevant also for *quantitative gnostical processes*, i.e. for processes of quantitative cognition of the real world. An approach presented here is coherent with physics but it is not a physical theory. To emphasize this, the author mostly presents only mathematical statements. Explanations and interpretations as well as applications are postponed to later publications.

A historical note is in order here. The first idea on possible connections between cybernetics and relativistic mechanics belongs to Guy Jumarie, who has been publishing a long series of papers on this subject since 1975. (See references in his book [1].) The same author rose a hypothesis on Minkowskian nature of the space of observation processes [2] but both his reasoning and results differ substantially from those presented below.

2. Main results

The space representing data has a Minkowskian metric. The process of obtaining real data (s.c. quantification) can be modelled by a group of pseudoeuclidian rotations of vectors on the Minkowskian plane. Transformations due to quantification uncertainty are subjected to a variational principle. Invariant of these transformations is the unknown true value of the quantity represented by uncertain data. A reasonable model of estimation is a group of Euclidian rotations which is completely dual to the quantifying group. Quantitative characteristics of uncertainty of data based on dissimilarity of events are obtained from which the information borne by each individual datum is derived. Each particular datum defines a gnostical cycle of quantification-estimation. For each closed gnostical cycle an ideal gnostical cycle exists which minimizes an unavoidable loss of information due to uncertainty.

3. Mathematical model of quantification

Quantification is a (measuring or counting) procedure which relates empirical quantities with real numbers.

Definitions. Let $\mathscr{E}$ be an empirical relational structure of empirical quantities q. Then the *ideal quantification* is a mapping $z_0: \mathscr{E} \leftrightarrow R_+$ where R_+ denotes the interval of positive real numbers z $(0 < z < \infty)$. The result $z_0(q_0)$ of ideal quantification of a quantity $q_0 \in \mathscr{E}$ will be denoted z_0 and called the *ideal value.*

For a full exposure of the theory of quantification see e.g. [3].

A result of an actual quantification of the quantity q_0 will not equal the ideal value z_0 in practice because of the influence of an uncertainty. This may result from imperfect realization of measuring or counting operations and/or from imperfect knowledge of the state of the quantified object. It is not necessary to describe the uncertainty statistically. We only assume that all contributions of uncertainty to a particular result of quantification may be characterized by a quantity q_u which is an element of an empirical relational structure.

Definitions. Let $\mathscr{F}$ be an empirical relational structure of quantities q_u, empirical characteristics of uncertainties. Then the *practical quantification* is a mapping $z: \mathscr{E} \times \mathscr{F} \leftrightarrow R_+$. A *possible datum* $z(q_0, q_u)$ is a result of a possible practical quantification, it will be denoted by a variable $z (z \in R_+)$. *Data,* results of practical quantification which actually took place, would be denoted by z_i (or by other indices $j, k, \ldots$). If F is an arbitrary function defined on R_+ then the symbol F_i denotes the value of the function $F(z)$ at the point $z = z_i$. ▮

AXIOM (Data model): Let z_0 be the ideal value. Let $\xi(q_u)$ be a result of the mapping $\xi: \mathscr{F} \leftrightarrow R_+$. Then the model of a possible datum is

$$z(q_0, q_u) = z_0(q_0)\xi(q_u). \quad ▮ \qquad (1)$$

The following parametrization of the effect of uncertainty is suitable:

$$z = z_0 e^{\Omega} \qquad (\Omega \in R_1)(z_0 \in R_+). \qquad (2)$$

Lemma 1. Let

$$\mathbf{u} := \begin{pmatrix} x \\ y \end{pmatrix} \qquad (3)$$

where

$$x := z_0 \operatorname{ch} \Omega \qquad (4)$$

$$y := z_0 \operatorname{sh} \Omega. \qquad (5)$$

Let

$$\mathbf{u}_0 := \begin{pmatrix} z_0 \\ 0 \end{pmatrix} \qquad (6)$$

and

$$\mathbf{K}_q(\Omega) := \begin{pmatrix} \operatorname{ch} \Omega & \operatorname{sh} \Omega \\ \operatorname{sh} \Omega & \operatorname{ch} \Omega \end{pmatrix}. \qquad (7)$$

Then

$$\mathbf{u} = \mathbf{K}_q(\Omega)\mathbf{u}_0. \quad ▮ \qquad (8)$$

A possible datum satisfies the equality

$$z = x + y. \qquad (9)$$

For another $\mathbf{u}' = \mathbf{K}_q(\Omega')\mathbf{u}_0$ we obtain from (7) and (8)

$$\mathbf{u}' = \mathbf{K}_q(\Omega' - \Omega)\mathbf{u}. \qquad (10)$$

Changing places of components x and y in (5), we get from (10) also

$${}_c\mathbf{u}' = \mathbf{K}_q(\Omega' - \Omega)\,{}_c\mathbf{u} \qquad (11)$$

where

$${}_c\mathbf{u} := \begin{pmatrix} y \\ x \end{pmatrix}. \qquad (12)$$

Definitions. Transformations $\mathbf{u} \to \mathbf{u}'$ (10) and ${}_c\mathbf{u} \to {}_c\mathbf{u}'$ (11) will be said to be *quantifying transformations.* Quantities $\mathbf{u}$ of type (3) will be called the *events of the first kind* and quantities ${}_c\mathbf{u}$ (12) together with

$${}^{\zeta}_{c}\mathbf{u} := \begin{pmatrix} -y \\ x \end{pmatrix} \qquad (13)$$

the *events of the second kind.* ▮

Theorem 1. Let ${}_1G_q$ and ${}_2G_q$ be sets of quantifying transformations (10) and (11), respectively. Then both sets are commutative groups with respect to the composition of transformations. Let G be the additive group of real numbers. Then all three groups are isomorphic. ▮

Proof. By verification that $\mathbf{K}_q(\Omega_1)\mathbf{K}_q(\Omega_2)=\mathbf{K}_q(\Omega_2)\mathbf{K}_q(\Omega_1)=\mathbf{K}_q(\Omega_1+\Omega_2)$, $\mathbf{K}_q(0)=\mathbf{I}$ and $\mathbf{K}_q^{-1}(\Omega)=\mathbf{K}_q(-\Omega)$. ∎

Quantification is thus modelled by groups ${}_1G_q$ and ${}_2G_q$.

As it is known, a Riemann's metric space is a variety on which a field of a metric tensor is defined which is symmetrical and complete. The scalar product of two arbitrary differentials $d\mathbf{u}'$ and $d\mathbf{u}''$ written in the matrix notation is given by the formula

$$(ds)^2 = d\mathbf{u}'^T \mathbf{g} d\mathbf{u}'', \tag{14}$$

where $\mathbf{g}$ is a matrix representation of the metric tensor.

Theorem 2. The unique metric on the variety of events invariant under the groups of quantifying transformations is the Minkowskian metric with the matrix

$$\mathbf{g}_M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad ∎ \tag{15}$$

Proof. Consider two arbitrary events $\mathbf{u}'^T=(z_0 \operatorname{ch} \Omega', z_0 \operatorname{sh} \Omega')$ and $(\mathbf{u}'')^T=(z_0 \operatorname{ch} \Omega'', z_0 \operatorname{sh} \Omega'')$ of the first kind. Their transforms are $\mathbf{K}_q(\Omega)\mathbf{u}'$ and $\mathbf{K}_q(\Omega)\mathbf{u}''$. The metric is invariant if the equality

$$d\mathbf{u}'^T \mathbf{g} d\mathbf{u}'' = d\mathbf{u}'^T \mathbf{K}_q^T(\Omega) \mathbf{g} \mathbf{K}_q(\Omega) d\mathbf{u}'' \tag{16}$$

holds identically for all pairs of events and for all $\mathbf{K}_q$. Denoting

$$\mathbf{g} = \begin{pmatrix} 1 & q \\ q & q' \end{pmatrix}$$

the metric matrix and substituting it into (16) we obtain the identity conditions $q=0$ and $q'=-1$. Considering the events ${}_c\mathbf{u}'$ and ${}_c\mathbf{u}''$ instead of events $\mathbf{u}'$ and $\mathbf{u}''$ we come to the same conclusion, only the sign of the scalar product changes,

$$d_c\mathbf{u}'^T \mathbf{g}_M d_c\mathbf{u}'' = -d\mathbf{u}'^T \mathbf{g}_M d\mathbf{u}''. \quad ∎ \tag{17}$$

For $\mathbf{u}'=\mathbf{u}''=\mathbf{u}$ we obtain from (14) after integration

$$s = z_0 = \sqrt{x^2 - y^2} \tag{18}$$

and for ${}_c\mathbf{u}'={}_c\mathbf{u}''={}_c\mathbf{u}$

$${}_cs = iz_0 \qquad (i=\sqrt{-1}) \tag{19}$$

as invariants of quantifying transformations. The quantities z_0 and iz_0 are thus pseudoeuclidian lengths of vectors $\mathbf{u}$ and ${}_c\mathbf{u}$, respectively. The effect of a practical quantification of a given quantity is thus equivalent to pseudoeuclidian rotations of vectors on Minkowskian plane. The events having the same length z_0 or iz_0 interpreted as points are lying on a Minkowskian circle. But it is important to consider also a more general case when variations of both variables Ω and z_0 are possible.

Definitions. Let

$$\kappa := \ln(z_0/z_{00}) \tag{20}$$

where z_{00} is a positive constant. Let

$$\lambda_{AB} := \int_{\widehat{AB}} \sqrt{d\mathbf{u}^T \mathbf{g}_M d\mathbf{u}/\mathbf{u}^T \mathbf{g}_M \mathbf{u}} = \int_{\widehat{AB}} \sqrt{(d\kappa)^2 - (d\Omega)^2} \tag{21}$$

where the line integral is taken along a path $\widehat{AB}$ between the points A and B of Minkowskian plane. The quantity λ_{AB} will be called *the relative length* of the path $\widehat{AB}$. The same definition will be used for ${}_c\mathbf{u}$. ∎

Theorem 3. Let A and B are some points (z_0, Ω_A) and (z_0, Ω_B), respectively. Then the relative length of the path $\widehat{AB}$ for which $\frac{d\kappa}{d\Omega}=0$ equals to

$${}_q\lambda_{AB} = i|\Omega_B - \Omega_A|. \qquad (i=\sqrt{-1}) \tag{22}$$

Its modulus represents a local maximum of moduli of all relative lengths of paths $\widehat{AB}$ obtained by small differentiable variations. ∎

Proof. Substituting the events $\mathbf{u}$ and ${}_c\mathbf{u}$ as functions of the coordinates (κ, Ω) into (21) we come for both kinds of events to the same formula

$$\lambda_{AB} = \left| \int_{\Omega_A}^{\Omega_B} \sqrt{\dot{\kappa}^2 - 1}\, d\Omega \right| \tag{23}$$

where $\dot{\kappa} = \frac{d\kappa}{d\Omega}$. This integral is a particular case of a more general one $\int_{\Omega_A}^{\Omega_B} F(\kappa, \dot{\kappa}, \Omega) d\Omega$ which has a stationary value under small smooth variations of the path between fixed points (Ω_A, F_A) and (Ω_B, F_B) if

$$\frac{\partial F}{\partial \kappa} - \frac{d}{d\Omega}\left(\frac{\partial F}{\partial \dot{\kappa}}\right) = 0. \tag{24}$$

This equation holds for a constant $\dot{\kappa}$. The stationary value is then

$$\lambda_{AB} = \sqrt{\dot{\kappa}^2 - 1}\, |\Omega_B - \Omega_A|. \tag{25}$$

Its modulus really has a local maximum (22) for $\dot{\kappa}=0$. Data thus correspond to points of a geodesic. ∎

4. Mathematical model of ideal estimation

Estimation is a numerical procedure which maps a collection of data into a quantity $\tilde{z}_0 \in R_+$ which is an *estimate* of the ideal value z_0. As shown above, the features of quantifying transformations are a consequence of positiveness of data. But estimating transformations are not yet determined. We shall require for them similar basic features as the quantifying transformations have: homogenity, double symmetry of the operator $\mathbf{K}_q$ and its uniform regularity (Det $\{\mathbf{K}_q\}=1$).

Theorem 4. Let ${}_1M_2$ be the variety of events $\mathbf{u}$ (3). Let $\mathbf{u}'^T:=(x'(x, y), y'(x, y))$ be a result of quantifying transformation (10). Then the group ${}_1G_q$ is the unique group of transformations ${}_1M_2 \to {}_1M_2$ which satisfies the following conditions:

I. Homogenity:

$$\frac{\partial x'}{\partial y}=0 \Rightarrow x'=x \tag{26}$$

$$\frac{\partial y'}{\partial x}=0 \Rightarrow y'=y. \tag{27}$$

II. Double symmetry:

$$\frac{\partial x'}{\partial x}=\frac{\partial y'}{\partial y}, \tag{28}$$

$$\frac{\partial y'}{\partial x}=\frac{\partial x'}{\partial y}. \tag{29}$$

III. Uniform regularity:

$$\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y}-\frac{\partial x'}{\partial y}\frac{\partial y'}{\partial x}=1. \quad \blacksquare \tag{30}$$

Proof. Transformation (10) clearly satisfies I. It follows further from (10) that $\frac{\partial x'}{\partial x}=\frac{\partial y'}{\partial y}=\operatorname{ch}\Omega$ and $\frac{\partial x'}{\partial y}=\frac{\partial y'}{\partial x}=\operatorname{sh}\Omega$. Conditions II and III are thus satisfied, too. Substitution of (28) and (29) into (30) gives the equations

$$\left(\frac{\partial x'}{\partial x}\right)^2-\left(\frac{\partial x'}{\partial y}\right)^2=1, \qquad \left(\frac{\partial y'}{\partial y}\right)^2-\left(\frac{\partial y'}{\partial x}\right)^2=1. \tag{31}$$

There exist exactly two pairs of solutions satisfying the condition of homogenity, the linear one coinciding with (10) and the quadratic one being equal to the following:

$$x'^2=x^2-y^2, \qquad y'^2=y^2-x^2 \tag{32}$$

But $y'^2<0$, therefore $(x', y') \notin {}_1M_2$. ∎

An analogous statement can be proved for the case of the events of the second kind, for the group ${}_2G_q$. We note that thanks to the double symmetry the transformation (11) results from (10) and inversely.

Looking for an estimating transformation dual to the quantifying one, we modify condition II:

$$\text{II}'. \qquad \frac{\partial x'}{\partial x}=\frac{\partial y'}{\partial y}, \qquad \frac{\partial x'}{\partial y}=-\frac{\partial y'}{\partial x}.$$

The condition of symmetry thus formally coincides with the Cauchy–Riemann conditions of analyticality of a complex function.

Theorem 5. Let R_2 be the 2-dimensional variety of events $\mathbf{u}^T=(x, y)$ and ${}_c\mathbf{u}^T=(y, x)$. Then the unique group of transformation $R_2 \leftrightarrow R_2$ satisfying conditions I, II′ and III is the group G_e of the transformations

$$\mathbf{u}'=\mathbf{K}_e(\omega)\mathbf{u}, \qquad {}_c\mathbf{u}'=\mathbf{K}_e(\omega){}_c\mathbf{u} \tag{33}$$

where

$$\mathbf{K}_e(\omega)=\begin{pmatrix}\cos\omega & -\sin\omega\\ \sin\omega & \cos\omega\end{pmatrix} \tag{34}$$

with a real parameter ω. ∎

Proof. The analogue of (31) resulting from II′ and III is

$$\left(\frac{\partial x'}{\partial x}\right)^2+\left(\frac{\partial x'}{\partial y}\right)^2=1, \qquad \left(\frac{\partial y'}{\partial y}\right)^2+\left(\frac{\partial y'}{\partial x}\right)^2=1. \tag{35}$$

Homogeneous linear solutions are

$$x'=Ax+By \qquad y'=Cx+Dy \tag{36}$$

where

$$A^2+B^2=1 \qquad C^2+D^2=1. \tag{37}$$

It results from II′ that $A^2=D^2$ and $B^2=C^2$. Then $A=\pm\cos\omega$ and $B=\pm\sin\omega$ where ω is a real parameter. The sign of B may be therefore chosen arbitrarily, we take $B=-\sin\omega$. From I′ we have $C=\sin\omega$. The sign of A and D is + because of (26). The linear solution is thus really (33).

Homogeneous quadratic solutions of (35) have the form

$$x'^2=x^2+y^2, \qquad y'^2=x^2+y^2. \tag{38}$$

This transformation has not an inverse for each event (x, y). It does not constitute a group. The group properties of transformation (33) are easily verifyable. ∎

Definition: For each $-\frac{\pi}{4}<\omega<\frac{\pi}{4}$, relation (33) will be called the *estimating transformation*. The group of these transformations will be denoted G_e. ∎

The three groups ${}_1G_q$, ${}_2G_q$ and G_e are thus the only groups satisfying all conditions I, III and a generalized condition of symmetry

$$\text{II}''. \qquad \frac{\partial x'}{\partial x}=\frac{\partial y'}{\partial y} \qquad \left|\frac{\partial x'}{\partial y}\right|=\left|\frac{\partial y'}{\partial x}\right|.$$

An analogue to Theorem 2 can be easily obtained: the unique metric on the variety R_2 invariant under the group G_e is the Euclidian metric with the metric matrix basic features as the quantifying transformations have: homogenity, double symmetry of the operator $\mathbb{K}_q$ and its uniform regularity (Det $\{\mathbb{K}_q\}=1$).

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An analogous statement can be proved for the case of the events of the second kind, for the group ${}_2G_q$. We note that thanks to the double symmetry the transformation (11) results from (10) and inversely.

Looking for an estimating transformation dual to the quantifying one, we modify condition II:

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The condition of symmetry thus formally coincides with the Cauchy–Riemann conditions of analyticality of a complex function.

Theorem 5. Let R_2 be the 2-dimensional variety of events $\mathbf{u}^T = (x, y)$ and ${}_\xi\mathbf{u}^T = (y, x)$. Then the unique group of transformation $R_2 \leftrightarrow R_2$ satisfying conditions I, II' and III is the group G_e of the transformations

$$\mathbf{u}' = \mathbb{K}_e(\omega)\mathbf{u}, \qquad {}_\xi\mathbf{u}' = \mathbb{K}_e(\omega)_\xi\mathbf{u} \tag{33}$$

where

$$\mathbb{K}_e(\omega) = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \tag{34}$$

with a real parameter ω. ▮

Proof. The analogue of (31) resulting from II' and III is

$$\left(\frac{\partial x'}{\partial x}\right)^2 + \left(\frac{\partial x'}{\partial y}\right)^2 = 1, \qquad \left(\frac{\partial y'}{\partial y}\right)^2 + \left(\frac{\partial y'}{\partial x}\right)^2 = 1. \tag{35}$$

Homogeneous linear solutions are

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This transformation has not an inverse for each event (x, y). It does not constitute a group. The group properties of transformation (33) are easily verifyable. ▮

Definition: For each $-\frac{\pi}{4} < \omega < \frac{\pi}{4}$, relation (33) will be called the *estimating transformation*. The group of these transformations will be denoted G_e. ▮

The three groups ${}_1G_q$, ${}_2G_q$ and G_e are thus the only groups satisfying all conditions I, III and a generalized condition of symmetry

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An analogue to Theorem 2 can be easily obtained: the unique metric on the variety R_2 invariant under the group G_e is the Euclidian metric with the metric matrix

$\mathbf{g}_E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The invariant of the group G_e is

$$r = \sqrt{x^2 + y^2}. \tag{39}$$

It equals to the radius of an Euclidian circle, to Euclidian length of events $\mathbf{u}$ rotated orthogonally by estimating transformations. The relative length of a path $\widehat{AB}$ between some points A and B defined analogously as in (21) with $\mathbf{g}_E$ instead of $\mathbf{g}_M$ is *minimal* if $r = \text{const}$, i.e. if the path corresponds to the estimating transformation. This minimal length is

$$_e\lambda_{AB} = |\omega_A - \omega_B|. \tag{40}$$

This statement can be verified in an analogous way as Theorem 3.

Lemma 2. Let $\mathbf{u}_i^T = (x_i, y_i) = (z_0 \operatorname{ch} \Omega_i, z_0 \operatorname{sh} \Omega_i)$ be an event corresponding to a datum z_i. Let

$$\tan \omega_i = -\operatorname{th} \Omega_i \tag{41}$$

and

$$\mathbf{u}_i'' = \begin{pmatrix} r_i \\ 0 \end{pmatrix}. \tag{42}$$

Then

$$\mathbf{u}_i'' = \mathbf{K}_e(\omega_i)\mathbf{u}_i. \quad ▮ \tag{43}$$

Proof. By substitution of (34), (41), (39) and (18) into (43). ▮

An analogous statement on the events of the second kind can be shown to have the form

$$_\xi\mathbf{u}_i'' = \mathbf{K}_e(\omega_i)_\xi\mathbf{u}_i \tag{44}$$

where $_\xi\mathbf{u}$ is (13).

If (41) holds for both kinds of events, then

$$x_i = r_i \cos \omega_i, \qquad y_i = -r_i \sin \omega_i. \tag{45}$$

5. Dissimilarities of events and their characteristics

Definitions. Two events $\mathbf{u}$ and $\mathbf{u}'$ are σ, ε-*similar* if

$$\frac{x}{y} = \sigma\left(\frac{x'}{y}\right)^{\varepsilon} \quad \text{where} \quad \sigma = \pm 1, \quad \varepsilon = \pm 1. \quad ▮ \tag{46}$$

If a couple of events, $\mathbf{u}$ and $\mathbf{u}'$, does not satisfy (46), then the events are dissimilar to an extent which is measurable by the difference of the quantities appearing on the left- and right-hand sides of (46) or by a quantity proportional to this difference. Using (4), (5) and (45) we obtain thus the normalized measures of dissimilarity of events summarized in Table 1.

Table 1. Measures of dissimilarity of events $\mathbf{u}$ and $\mathbf{u}'$

Types of similarity		Measure of dissimilarity	
σ	ε	Minkowskian	Euclidian
+1	+1	$\operatorname{sh}(\Omega' - \Omega)$	$\sin(\omega' - \omega)$
+1	−1	$\operatorname{ch}(\Omega' - \Omega)$	$\cos(\omega' + \omega)$
−1	+1	$\operatorname{sh}(\Omega' + \Omega)$	$\sin(\omega' + \omega)$
−1	−1	$\operatorname{ch}(\Omega' + \Omega)$	$\cos(\omega' - \omega)$

Such characteristics appear with composed transformations due to products $\mathbf{K}_q(\Omega')\mathbf{K}_q(\Omega)$, $\mathbf{K}_q(\Omega')\mathbf{K}_q^{-1}(\Omega)$, $\mathbf{K}_e(\omega')\mathbf{K}_e(\omega)$ and $\mathbf{K}_e(\omega')\mathbf{K}_e^{-1}(\omega)$. Special cases $\mathbf{K}_q^2(\omega) \equiv \mathbf{K}_q(2\omega)$ and $\mathbf{K}_e^2(\omega) \equiv \mathbf{K}_e(2\omega)$ are worth to be analysed in more details. They characterize the quality of a possible datum.

Theorem 6. Let ω be such that $\operatorname{th} \Omega = -\tan \omega$. Let

$$\xi := z/z_0, \tag{47}$$

and f, h_e be such that

$$\mathbf{K}_e^2(\omega) = \begin{pmatrix} f & -h_e \\ h_e & f \end{pmatrix}. \tag{48}$$

Let

$$h_q := \operatorname{sh} 2\Omega. \tag{49}$$

Then

$$f = \frac{x^2 - y^2}{x^2 + y^2} = \frac{2}{\xi^2 + \xi^{-2}} = \frac{1}{\operatorname{ch} 2\Omega} = \cos 2\omega \tag{50}$$

$$h_q = \frac{2xy}{x^2 - y^2} = \frac{\xi^2 - \xi^{-2}}{2} = \operatorname{sh} 2\Omega = -\tan 2\omega = -\frac{h_e}{f} \tag{51}$$

$$h_e = \frac{-2xy}{x^2 + y^2} = -\frac{\xi^2 - \xi^{-2}}{\xi^2 + \xi^{-2}} = -\operatorname{th} 2\Omega = \sin 2\omega = -h_q f \tag{52}$$

$$\mathbf{K}_q^2(\Omega) = \begin{pmatrix} 1/f & h_q \\ h_q & 1/f \end{pmatrix}. \quad ▮ \tag{53}$$

Proof. Using (2), (4), (5), (7), (35), the assumptions of Theorem 6 and the formulae of the elementary functions. ▮

6. Information of a datum

Let $f \equiv f(x, y)$ be given by (50). Consider scalar fields of $1/f(x, y)$ over the Minkowskian plane M_2 and of $f(x, y)$ over the Euclidian plane E_2. Laplace's operators $\nabla^2(\cdot) = \operatorname{div} \operatorname{grad}(\cdot)$ are, as known, $\nabla_M^2(\cdot) := \frac{\partial^2(\cdot)}{\partial x^2} - \frac{\partial^2(\cdot)}{\partial y^2}$ and $\nabla_E^2(\cdot) := \frac{\partial^2(\cdot)}{\partial x^2} + \frac{\partial^2(\cdot)}{\partial y^2}$.

Theorem 7. Let (x, y) be an arbitrary point of the variety R_2, for which $x \neq |y|$. Then

$$\nabla_M^2\left(\frac{1}{f}\right) + \frac{4}{x^2-y^2}\,\frac{1}{f} = 0 \tag{54}$$

and

$$\nabla_E^2(f) + \frac{4}{x^2+y^2}\,f = 0. \tag{55}$$

Proof. By application of Laplace's operators to (50). ∎

Definitions: For each complex $p \neq 0$, $p \neq 1$ let

$$H(p) = -p\ln(p) - (1-p)\ln(1-p) \tag{56}$$

where the main value of $\ln(\cdot)$ is considered.

Let

$$p_q = (1+ih_q)/2 \qquad (i=\sqrt{-1}) \tag{57}$$

$$p_e = (1+h_e)/2. \tag{58}$$

Then the quantities

$$I_q = H(1/2) - H(p_q) \tag{59}$$

$$I_e = H(1/2) - H(p_e) \tag{60}$$

will be called the *quantifying and estimating change of information*, respectively.

Both quantities I_q and I_e are functions only of the argument z/z_0, as seen from (57), (58), (47), (51) and (52). They should characterize the changes of information on the ideal value z_0 borne by the datum z.

Theorem 8. Let th $\Omega = -\tan\omega$. Then

$$I_q = 2\int_0^{h_q} \omega\, dh_q \tag{61}$$

and

$$I_e = 2\int_{h_e}^{0} \Omega\, dh_e. \tag{62}$$

∎

Proof. Using equalities $2\omega = -\arctan(h_q)$ and $2\Omega = -\text{arcth}(h_q)$ (see (51) and (52), respectively) we obtain from (61) and (62) the integrals

$$I_q = -h_q\,\text{arcth}(h_q) + \ln\sqrt{1+h_q^2} \tag{63}$$

and

$$I_e = h_e\,\text{arctan}(h_e) + \ln\sqrt{1-h_e^2} \tag{64}$$

which after substitution from (56)–(58) equal to (59) and (60). ∎

We note that for $\Omega \neq 0$

$$I_q < 0 \quad\text{and}\quad I_e > 0. \tag{65}$$

Theorem 9. Let $x', y' \in R_2$ be such that $x'^2 + y'^2 = r^2 = \text{const}$. Then

$$\left[\nabla_M^2\left(\frac{1}{f}\right)\right]_{x=x',y=y'} = c_1\left[\frac{d^2 I_e}{dh_e^2}\right]_{x=x',y=y'}, \tag{66}$$

where c_1 is a constant.

Let $x', y' \in R_2$ be such that $x'^2 - y'^2 = z_0^2 = \text{const}$. Then

$$[\nabla_E^2(f)]_{x=x',y=y'} = c_2\left[\frac{d^2 I_q}{dh_q^2}\right]_{x=x',y=y'}, \tag{67}$$

where c_2 is a constant. ∎

Proof. The second term in (54) may be rewritten as $4f^{-2}/(x^2+y^2)$. Using the identity $f^2 = 1-h_e^2$ (see (50) and (52)) together with (62) we come to $c_1 = -4/r^2$. The proof of (67) is analogous. ∎

Both changes of information are thus dependent on the dissimilarity characteristic f of the event.

7. Relation between two changes of information

Theorem 10. Let th $\Omega_i = -\tan\omega_i$ and let $\Omega_i \neq 0$. Then

$$I_{qi} + I_{ei} < 0. \tag{68}$$

∎

Proof. From (51), (52), (61) and (62) using (I.47)

$$\left|\frac{dI_q}{dh_q}\,\frac{dh_q}{d\xi}\right| = 2|\omega|(\xi+\xi^{-3}) \quad \left|\frac{dI_e}{dh_e}\,\frac{dh_e}{d\Omega}\right| = 2|\Omega|\,\frac{8\xi^{-1}}{(\xi^2+\xi^{-2})^2}.$$

Consider integral

$$2|\omega| = \left|\int_0^{h_q} \frac{dh}{1+h^2}\right| = \left|\int_1^{\xi} \frac{4}{\xi^2+\xi^{-2}}\,\frac{d\xi}{\xi}\right|$$

which taken per partes gives

$$\left|\frac{4}{\xi^2+\xi^{-2}}\ln\xi + 8\int_1^{\xi}\frac{(\xi-\xi^{-3})\ln\xi}{(\xi^2+\xi^{-2})^2}\,d\xi\right|.$$

The latter term is positive for all $\xi \in R_+$, therefore $2|\omega| = \left|\frac{4}{\xi^2+\xi^{-2}}\ln\xi + c^2\right|$ with a $c^2 > 0$. Then the ratio is

$$\left|\frac{dI_q}{d\xi}\bigg/\frac{dI_e}{d\xi}\right| = |(\xi^2+\xi^{-2})^2/4 + c^2(\xi^2+\xi^{-2})^3/16\ln\xi|.$$

For all $\xi > 1$ is thus $\left|\frac{dI_q}{d\xi}\right| > \left|\frac{dI_e}{d\xi}\right|$. Taking into account (65) we obtain (68). But both I_q and I_e are symmetrical functions of ξ and ξ^{-1}, therefore the same relation holds for $\xi < 1$. ∎

Theorem 11. Let $z_i = z_0 e^{\Omega_i}$ be a datum. Let us consider a path $\widehat{AB}$ between the points $A := (z_0, 0)$ and $B := (z_0 \operatorname{ch} \Omega_i,\ z_0 \operatorname{sh} \Omega_i)$ obtained by small differentiable variations from the path $(\widehat{AB})_q$ for which $\kappa = \text{const}$. Let I_{ei} be the estimating change of information determined by the z_i. Let δI_{ei} be a variation due to variation of the path $\widehat{AB}$. Then

$$\delta I_{ei} \leqq 0 \tag{69}$$

and the path $(\widehat{AB})_q$ maximizes the quantity I_{ei}. ∎

Proof. The relative length λ_{AB} of the path $\widehat{AB}$ satisfies the inequality

$$|\Omega| \geqq |\lambda_{AB}| \tag{70}$$

as follows from Theorem 3. Because of the relation $\operatorname{sign}(\Omega) = -\operatorname{sign}(h_e)$ we may write (62) as

$$I_e = 2 \int_0^{|h_{ei}|} |\Omega| d|h_e|. \tag{71}$$

Therefore

$$I_{ei} \geqq 2 \int_0^{|h_{ei}|} |\lambda_{AB}| d|h_e|$$

for all variations of the mentioned type. ∎

A quantifying transformation corresponds thus to a path which maximizes the estimating change of information. It may be shown analogously that the estimating transformation corresponds to a path which maximizes the quantifying change of information I_{qi} which is negative:

$$\delta I_{qi} \leqq 0. \tag{72}$$

8. A quadruple of mutually complementary groups of gnostical transformations

The groups G_q and ${}_2G_q$ of quantifying transformations and the group G_e of estimating transformations are mutually dual, but they do not represent a system which would be in a sense complete. Therefore a further group is to be introduced.

Definitions. Let **u** and **u**′ be arbitrary events ($\mathbf{u}, \mathbf{u}' \in R_2$). Then the transformation

$$\mathbf{u}' = \mathbf{K}_a(k)\mathbf{u} \qquad (k \in R_+) \tag{73}$$

where

$$\mathbf{K}_a(k) = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \tag{74}$$

is the *attenuating/amplifying transformation.*

Under *gnostical* we shall understand quantifying, estimating and/or attenuating/amplifying. ∎

The set G_a of all possible transformations $\mathbf{K}_a(k)$ is obviously a commutative group. Both relative lengths Ω and ω are invariants under the group G_a. It means that the uncertainty of an event cannot be changed by attenuation or amplification. Therefore, if I_a denotes the attenuating/amplifying change of information then the identity

$$I_a \equiv 0 \tag{75}$$

holds.

We came to four commutative groups of transformations, summarized in Table 2.

Table 2. Four commutative groups of gnostical transformations

Group		Operator	Para-	Transfor-	Inva-	Variable
Symbol	Name	**K**	meter	mations	riant	
${}_1G_q$, ${}_2G_q$	quantificating	$\mathbf{K}_q(\Omega)$	Ω		$\sqrt{x^2-y^2}$	$\sqrt{x^2+y^2}, \frac{y}{x}$
G_e	estimating	$\mathbf{K}_e(\omega)$	ω	$\mathbf{u}' = \mathbf{K}\mathbf{u}$, ${}_e\mathbf{u}' = \mathbf{K}_e\mathbf{u}$	$\sqrt{x^2+y^2}$	$x^2-y^2, \frac{y}{x}$
G_a	attenuating / amplificating	$\mathbf{K}_a(k)$	k		$\frac{x}{y}$	$\sqrt{x^2-y^2}, \sqrt{x^2+y^2}$

As demonstrated in Table 2, the four groups of channels are mutually complementary. Their usefulness is manifested especially by their suitability to create an important closed cycle of gnostical transformations.

9. The ideal gnostical cycle and its optimality

Definition. Ideal gnostical cycle (IGC)$_i$ defined by a datum z_i ($z_i \neq z_0$) which has a model $z = z_0 e^{\Omega_i}$ is a triple of continuous segments of lines interconnecting successively three points of the variety R_2 determined by events $\mathbf{u}_i$ (3) and $\mathbf{u}_i''$ (42) (or corresponding points ${}_e\mathbf{u}_0$, ${}_e\mathbf{u}_i$ and ${}_e\mathbf{u}_i''$, respectively). The segments are: a segment of the Minkowskian circle with the radius z_0 (18), that of the Euclidian circle with the radius r_i (39) and a segment of a straight line.

The IGC is thus a composition of effects of three groups ${}_1G_q$, G_e and G_a or ${}_2G_q$, G_e and G_a. It should be a closed cycle, therefore $\tan \omega_i = -\text{th}\, \Omega_i$ is assumed together with the requirement

$$k_i = z_0/r_i \tag{76}$$

which defines the closing transformations. The IGC includes not only the defining (end) points of the three segments but also all inner points of these segments. A graphical representation of an IGC is in Fig. 1.

Closeness of a gnostical cycle corresponds to the requirement that the result of ideal estimation should be equal to the starting (unknown, ideal) value of the quantity which is the object of quantification. Idealness of the IGC has two important aspects:

(1) A gnostical cycle cannot be closed on practice because it would require the exact knowledge of the particular uncertainty borne by a datum.

(2) The IGC is the best possible closed cycle between all cycles passing the same defining (end) points, as shown below.

Main theorem. Let $(\text{IGC})_i$ be the ideal gnostical cycle determined by a datum z_i and by its ideal value z_0. Let $(\text{VC})_i$ be a closed cycle passing through the points $(z_0, 0)$

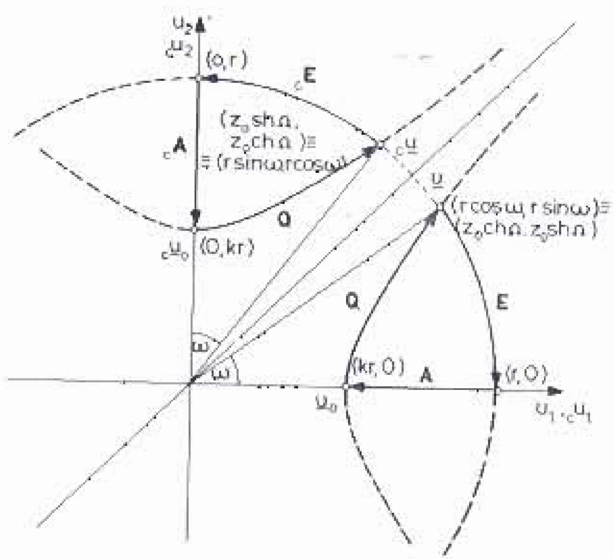

Fig. 1. The ideal gnostical cycle of an event **u** of the first kind and of an event ${}_c$**u** of the second kind. Q — quantification, E — estimation, A — attenuation, ${}_cQ$, ${}_cE$, ${}_cA$ — analogous for the event ${}_c$**u**

and (x_i, y_i), obtained from $(\text{IGC})_i$ by small smooth variations. Let $z_i \neq z_0$. Let I denote successively I_q, I_e and I_a. Then the $(\text{IGC})_i$ is optimal in the sense that

$$\oint_{(\text{VC})_i} dI \leq \oint_{(\text{IGC})_i} dI < 0. \quad ▮ \tag{77}$$

Proof. The overall change of information I_c within the ideal gnostical cycle is given by a sum

$$I_c = I_q + I_e + I_a. \tag{78}$$

But $I_a \equiv 0$ by (75). The sum $I_q + I_e$ is negative by Theorem 10 and it represents a maximum of such quantities obtained by small variations of the paths in the sense of Theorem 11 and of (69) and (72). ▮

The knowledge of the ideal gnostical cycle of each individual datum enables us to attack effectively the problem of optimum treatment of small data samples.

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Гностическая теория отдельных действительных данных

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(Прага)

В статье изложены теоретические основы нового подхода к проблеме переработки действительных данных. В отличие от статистической, предлагаемая «гностическая» теория основана на детальном изучении общих закономерностей преобразований индивидуальных данных под влиянием неопределенностей. Из единственной простой аксиомы выводятся геометрические свойства пространства, в котором изображаются действительные данные. Доказывается, что количественные характеристики неопределенности данных удовлетворяют уравнению диффузии, из которого выводится формула для количества информации, содержимой в одном отдельном элементе набора действительных данных. Каждое из этих данных определяет некоторый идеальный замкнутый цикл гностических преобразований, минимализирующий неизбежную потерю информации, под влиянием неопределенности.

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